
Human Footfall Induced Vibrations

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MAVERICK UNITED CONSULTING ENGINEERS

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LIST OF SYMBOLS AND NOTATIONS

Elemental Notations

$\{y\}$	=	displacement function
$[N]$	=	shape functions matrix
$\{f\}$	=	element forces in element axes including fixed end forces
$\{d\}$	=	element deformation in element axes
$\{b\}$	=	element body forces
$[k]$	=	element constitutive matrix
$[m]$	=	element mass matrix
$[c]$	=	element viscous damping matrix
$\{p\}$	=	element nodal loading vector
$[T]$	=	transformation matrix
W	=	work done by external loads

SDOF, MDOF and Modal Dynamic Equation of Motion Notations

m	=	SDOF mass
$[M]$	=	Global MDOF mass matrix
c	=	SDOF viscous damping constant
$[C]$	=	Global MDOF viscous damping matrix
k	=	SDOF stiffness
$[K]$	=	Global MDOF stiffness matrix
u	=	SDOF displacement
$\{u\}, \{U\}$	=	Global MDOF displacement matrix
$\{P\}$	=	Global nodal loading vector
$M_i, [M]$	=	Modal (generalized) mass and modal (generalized) mass matrix
$C_i, [C]$	=	Modal (generalized) damping and modal (generalized) damping matrix
$K_i, [K]$	=	Modal (generalized) stiffness and modal (generalized) stiffness matrix
$\xi_i, \{\xi_i\}$	=	Modal displacement response and modal displacement response vector

SDOF Dynamic Notations

ω_n	=	Natural circular frequency, $(k/m)^{1/2}$
ω_d	=	Damped natural circular frequency, $\omega_n(1-\zeta^2)^{1/2}$
ω	=	Frequency of forcing function
c	=	Viscous damping constant
c_{cr}	=	Critical viscous damping constant, $2(km)^{1/2} = 2m\omega_n$
ζ	=	Damping ratio (fraction of critical), c/c_{cr}
δ	=	Logarithmic decrement

SDOF Free Vibrational Notations

G	=	Complex starting transient response function, $G = G_R + iG_I$
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SDOF Time Domain Loading and Transient and Steady-State Response Notations

$P(t)$	=	Loading function
p_0	=	Force excitation amplitude
p_0/k	=	Static displacement
$D(t)$	=	Dynamic amplification factor

D_{\max}	=	Maximum dynamic amplification factor
$u(t)$	=	Displacement response, $D(t)(p_0/k)$
u_{\max}	=	Maximum displacement response, $D_{\max}(p_0/k)$

Modal Time Domain Loading and Transient and Steady-State Notations

$\{P(t)\}$	=	Loading function vector
$P_i(t)$	=	Modal loading function, $P_i(t) = \{\phi_i\}^T \{P(t)\}$
ρ_{0i}	=	Modal force excitation amplitude
ρ_{0i}/K_i	=	Modal static displacement
$D_i(t)$	=	Modal dynamic amplification factor
$D_{i \max}$	=	Modal maximum dynamic amplification factor
$\xi_i(t)$	=	Modal displacement response, $\xi_i(t) = D_i(t)\rho_{0i}/K_i$
$\xi_{i \max}$	=	Modal maximum displacement response, $\xi_{i \max} = D_{i \max} \rho_{0i}/K_i$
$\{u(t)\}$	=	Displacement response vector, $\{u(t)\} = [\Phi]\{\xi(t)\}$

SDOF Frequency Domain Loading and Steady-State Response Notations

$P(t)$	=	SDOF Time domain harmonic loading function, $P(t) = \text{Real} [P(\omega)e^{i\omega t}]$
$P(\omega)$	=	SDOF frequency domain complex harmonic loading function
p_0	=	SDOF harmonic loading amplitude
p_0/k	=	SDOF static displacement
$D(\omega)$	=	SDOF (magnitude of the) dynamic amplification factor
D_{resonant}	=	SDOF (magnitude of the) dynamic amplification factor at resonance when $\omega = \omega_n$
D_{\max}	=	SDOF maximum (magnitude of the) dynamic amplification factor when $\omega = \omega_n(1-2\zeta^2)^{1/2}$
$F(\omega)$	=	SDOF complex displacement response function (FRF), $F(\omega) = D(\omega)(p_0/k)e^{-i\theta}$
$H(\omega)$	=	SDOF transfer function, $H(\omega) = D(\omega)(1/k)e^{-i\theta}$
F_{resonant}	=	SDOF complex displacement response function at resonance, $F_{\text{resonant}} = D_{\text{resonant}}(p_0/k)e^{-i\theta}$
F_{\max}	=	SDOF complex maximum displacement response function, $F_{\max} = D_{\max}(p_0/k)e^{-i\theta}$
$u(t)$	=	SDOF time domain displacement response, $u(t) = \text{Real} [F(\omega)e^{i\omega t}]$
u_{resonant}	=	SDOF time domain displacement response at resonance, $u(t) = \text{Real} [F_{\text{resonant}} e^{i\omega t}]$
u_{\max}	=	SDOF time domain maximum displacement response, $u(t) = \text{Real} [F_{\max} e^{i\omega t}]$
T_r	=	SDOF transmissibility of displacement, acceleration or force

Modal Frequency Domain Loading and Steady-State Response Notations

$\{P(t)\}$	=	Time domain harmonic loading function vector, $\{P(t)\} = \text{Real} [\{P(\omega)\} e^{i\omega t}]$
$\{P(\omega)\}$	=	Frequency domain complex harmonic loading function vector
$P_i(\omega)$	=	Modal frequency domain complex harmonic loading function vector, $P_i(\omega) = \{\phi_i\}^T \{P(\omega)\}$
ρ_{0i}	=	Modal harmonic loading amplitude
ρ_{0i}/K_i	=	Modal static displacement
$D_i(\omega)$	=	Modal (magnitude of the) dynamic amplification factor
$D_{i \text{ resonant}}$	=	Modal (magnitude of the) dynamic amplification factor at resonance when $\omega = \omega_{ni}$
$D_{i \max}$	=	Modal maximum (magnitude of the) dynamic amplification factor when $\omega = \omega_{ni}(1-2\zeta_i^2)^{1/2}$
$\xi_i(\omega)$	=	Modal complex displacement response function (FRF), $\xi_i(\omega) = D_i(\omega)\rho_{0i}/K_i e^{-i\theta_i}$
$\xi_{i \text{ resonant}}$	=	Modal complex displacement response function at resonance, $\xi_{i \text{ resonant}} = D_{i \text{ resonant}} \rho_{0i}/K_i e^{-i\theta_i}$
$\xi_{i \max}$	=	Modal complex maximum displacement response function, $\xi_{i \max} = D_{i \max} \rho_{0i}/K_i e^{-i\theta_i}$
$\{u(t)\}$	=	Time domain displacement response vector, $\{u(t)\} = \text{Real} [[\Phi]\{\xi(\omega)\}e^{i\omega t}]$

Additional Abbreviations

ML: Materially Linear
MNL: Materially Nonlinear
GL: Geometrically Linear
GNL: Geometrically Nonlinear
[] = matrix
{ } = column vector
< > = row vector

1.1 GL, ML Human Footfall Induced Vibrations ¹

1.1.1 Limitations

The proposed method does not address the analysis of very lightweight floors (e.g. domestic timber floors) where the interaction between the person walking on the floor and the floor itself is more complex, and cannot be treated simply as an applied dynamic load.

1.1.2 Finite Element Mesh and Model Parameters

As with any dynamic response problem, if only the first few resonance frequencies and mode shapes are required, then a relatively coarse mesh can be used. If higher frequencies are required, then the mesh must be suitably fine to model the more complicated mode shapes. A good parameter may be the modal mass. We know that the modal mass of **all** the bending modes

of a **beam** (single spanning slab) is **half** the total beam (single spanning slab) mass if **simply supported**,
of a **beam** is **quarter** the beam mass if a **cantilever**, and
of a **plate** is **quarter** the plate mass if **simply supported**.

When higher modes deviate from the correct modal mass value, then we know that the mesh cannot capture the mode shape. With more complicated structures, an h- or p-refinement should be performed and the **modal masses of the original and the refined mesh should be compared**; modal masses of higher modes which begin to show discrepancies indicates the extent of the capability of the mesh in capturing these higher modes. The following modal analyses of a simply supported 10m UB457x191x98 lumped mass beam model were performed in OASYS/GSA 8.0.15. We shall describe only the bending modes (not axial). The total mass of the structure is 981.3kg, hence the correct modal mass is 490.65kg. It is usually recommended that there be 4 elements (or 2^n , n integer) for each half sine of mode. As the number of elements in the odd number models increases, gradually, the modal mass becomes more accurate.

Beam Mesh	Natural Frequencies (Hz)	Modal Mass (kg)	Comment
2 Elements	15.03	490.6	
3 Elements	15.14, 55.87	All 654.2	Very poor
4 Elements	15.16, 57.52, 113.3	All 490.6	Good for first mode
5 Elements	15.17, 58.19, 119.9, 179.4	All 542.4	Very Poor
6 Elements	15.17, 58.38, 122.2, 192.7, 249.5	490.6, 654.2, 490.6, 654.2, 490.6	Very Poor
7 Elements	15.17, 58.48, 123.3, 198.5, 269.7, 320.9	All 516.2	Poor
8 Elements	15.17, 58.53, 123.9, 201.6, 280.2, 347.3, 392.6	All 490.6	Good for first 2 modes

With floor problems, because of the high repetition of the structure, there is likelihood that there will be many modes of similar frequency. Symmetry boundary conditions may be argued, but they will only capture odd bending modes. Antisymmetric boundary conditions are required for even modes. Combination is quite impractical in dynamic response studies. Moreover, the symmetric or antisymmetric boundary conditions only capture the stiffness of the omitted structure and the correct natural frequencies. However, for response calculations, the omitted structure will result in lower modal masses if nothing else is done, and hence a higher response. Hence, the mass associated with the truncated portion of the modes have to be lumped onto the boundary to bring the modal mass up to that of the original model and hence produce the correct response. Because of these unnecessary complications, a full model is strongly recommended for dynamic analyses.

¹ WILLFORD, Michael & YOUNG, Peter. *Towards A Consistent Approach of Predicting Footfall-Induced Structural Vibrations*. Advanced Technology Ove Arup & Partners International Ltd., London.

Composite floors of concrete slab and steel beams if acting compositely (i.e. has sufficient shear studs for full composite action) can be modeled with shells for the slab and compatible beam elements for the beams. Offset elements should be used to model the offset between the elastic neutral axes of the slab and the beams. In these cases of composite floors, the shell formulation must include in-plane stiffness (on top of bending stiffness). If the floor has a layer of screed and/or stone finish on top of the concrete slab, the mass and stiffness should be included and the offset from the elastic neutral axes of the slab + finishing to that of the beam should be used. Connections between elements should be at their neutral axis. Connections of slabs with offset beams are no different, the elastic neutral axis should be found and both the beams and the slabs should be offset from their common neutral axis to connect into column (beam) elements. This is important especially so in linear dynamic problems where an incorrect connection point will result in incorrect mode shapes.

The recommended dynamic parameters for steel and concrete are as follows.

Material	Density (tonnes/m ³)	Young's Modulus (kN/m ²)	Poisson's Ratio
NW concrete	2.40	38 x 10 ⁶	0.2
LW concrete	1.88	22 x 10 ⁶	0.2
Steel	7.84	205 x 10 ⁶	0.3

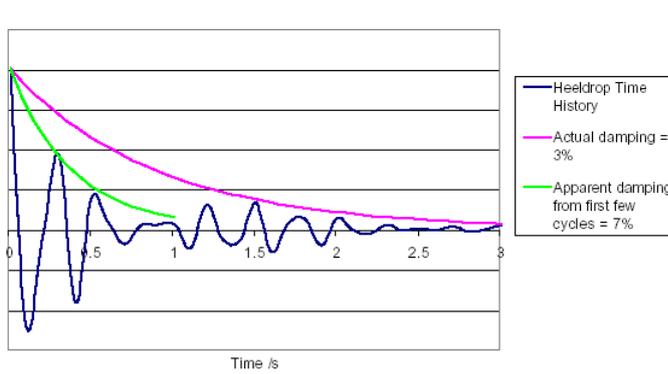
The calculation of reinforced concrete section properties can often assume the concrete is uncracked, but the nominal extreme fibre stresses should be estimated under in-service loads to validate this assumption. Normally the effect of cracking will be negligible if extreme fibre stresses are less than 2N/mm². The objective is to obtain the best estimate of the natural frequency, and when cracking is extensive, judgment is required to assess the most representative section properties to assume. The section properties so derived should be based on dynamic E values and allow for tension zone stiffening. If it is expected that the concrete may be cracked then a reduced Young's modulus should be used. Concrete screeds, when they are present, can be included as contributing to dynamic stiffness.

At the very low strain levels associated with footfall vibration it can usually be assumed that structural elements are fully continuous and behave in a fully composite manner unless connections are detailed specifically to prevent this (e.g. true pin connection, truss supported at top chord only). Normally for strength or serviceability design structural engineers will assume pinned end connections. For dynamics assume all connections are fixed unless they truly are pins. This is due to the very small strains involved with dynamics. An external wall or façade can generally be assumed to offer a continuous line of support to the floor edge. If you have a façade around your building it may offer vertical support to your floor. Due to the small strains involved with dynamics provide vertical restraint at façade locations. The same applies around concrete core walls. Columns offer rotational stiffness to floor beams assuming full continuity as well as vertical support. If modelling a floor plate with pin supports at column positions the rotational stiffness of the beams can be increased by removing the pin supports and modelling the columns above and below the floor. The bending stiffness of the columns will help increase the natural frequency of the floor.

There is also the danger of over restraining. If modelling a floor plate using 2D elements and offset beams, make sure that the slab edges are not fully restrained against moving laterally (i.e. that it is pinned). Otherwise, this will result in the slab attracting membrane forces and will give a higher natural frequency than the floor actually has.

The best estimate of the in-service mass should be used. The actual mass of fit-out in a building is often significantly less than the design allowances, and actual live loads in offices are often only 10% of the design value, and are negligible on footbridges. Typically the non-structural mass for an office floor can be taken as 100kg/m². Do NOT apply the full live load as added mass. For stairs and bridges NO live load should be added. It should be noted that over-estimating the mass can often result in unconservative predictions of dynamic response. For trapezoidal decking, you may need to modify the density of the 2D elements to take account of the large voids in trapezoidal decking. It is vital that mass is correctly modeled to calculate the right natural frequencies and modal masses.

Damping values quoted in the literature for floors have often been estimated from heel-drop tests and caution must be used with these values. The damping so obtained is usually not a true measure of modal damping because several modes are responding together, and the initial decay of a heel drop includes transfer of energy between modes as well as dissipation. The apparent damping so obtained is too high.



It has been seen that for composite structures the effective damping tends to decrease as the span increases. This may be due to the greater size of the steel elements (which have lower intrinsic damping than concrete) relative to the concrete in longer spans.

	Bare Structures		Fitted Out Floors
	mean	cov	
Composite steel/concrete and pre-stressed concrete	$(2.5 - 0.046x)\%$ where $x = \text{span in m}$	0.3	1.5 to 4.0%
Reinforced concrete	2.0%	0.3	2.0 to 5.0%

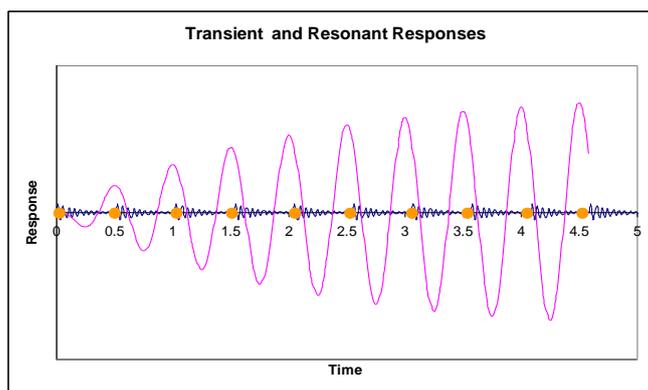
For a typical composite office floor or reinforced concrete floor with partitions and raised floors damping of 3% of critical may be assumed. For a composite floor with no furniture, partitions, raised floors (e.g. airport terminal) damping may be as low as 1.0% of critical damping.

1.1.3 (Non-Crowd or Asynchronous) Human Induced Vertical Footfall Vibrations from Walking

This is evaluated for a single person walking. If several people are walking together, unless they are marching in a disciplined manner, the footfall rates and phasing will not be coordinated, and hence the response will not increase in proportion to the number of people. The response due to a single person walking is most when the structure is empty as to minimize the modal mass and not add additional human damping. It has been observed that the damping of a structure increases if a large number of people are present on the structure. The human body can be represented as a dynamic system having two or more main degrees of freedom.

1.1.3.1 Dynamic Response To Vertical Dynamic Forces From Walking on Flat and on Stairs

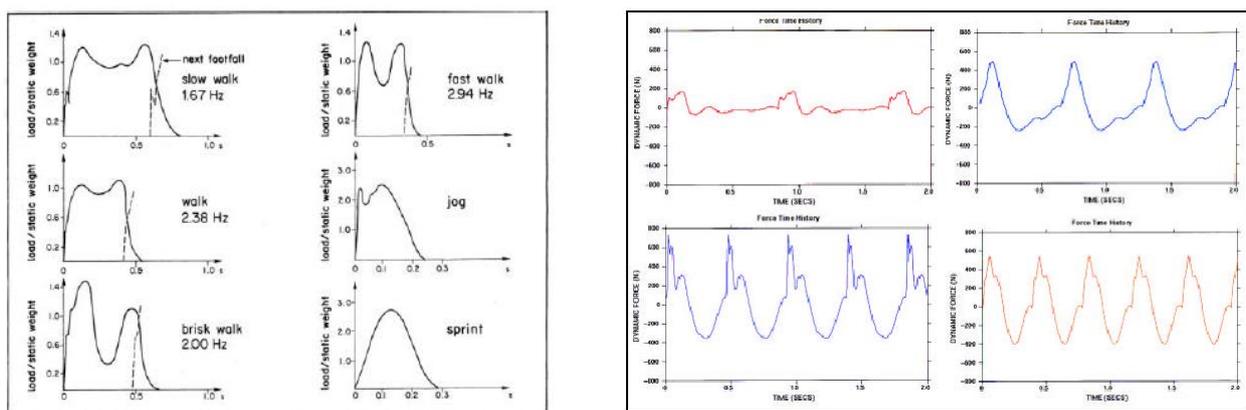
Depending on the relative frequencies of harmonic excitation and the natural frequencies of the structure, a different response will be obtained. The following graph shows a 2Hz and a 26Hz floor being excited by 2Hz walking. There is clear resonant build-up in the first case and clear individual transient response in the second case.



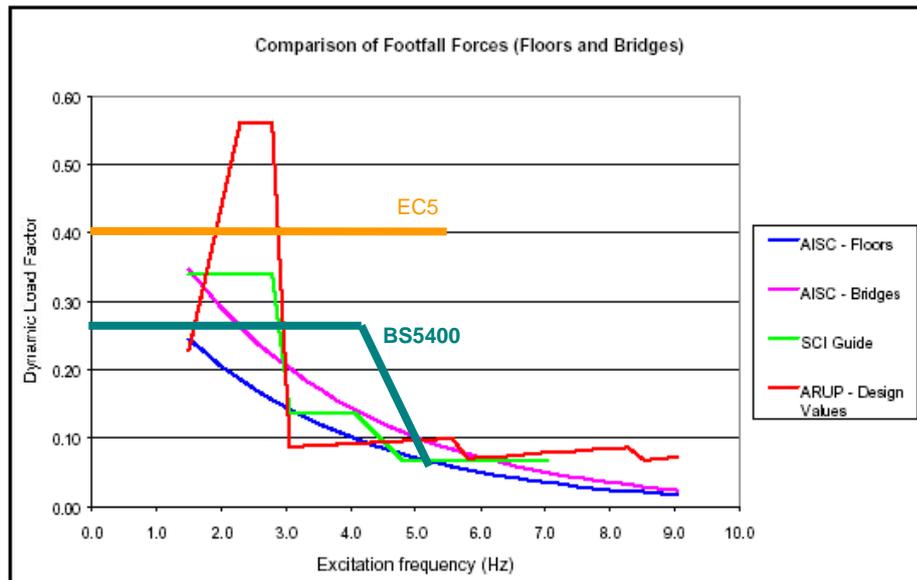
1.1.3.1.1 Resonant Excitations For Modes Up To 15Hz (Flat) and 22Hz (Stairs)

In this LINEAR FREQUENCY DOMAIN solution, not only that the static response has to be added separately, but also the mean of the dynamic excitation has also got to be added separately as a static response. This is because the mean of the dynamically applied force is not included in the dynamic excitations. **Hence the total response in this frequency domain dynamic analysis = static response to mean of dynamic excitation + dynamic response + static response to static loads.** However, usually in footfall induced response, it is not the total response which is of importance, as the criteria is not from a structural safety viewpoint, but rather from a human comfort viewpoint, the latter of which is measured in terms of acceleration and velocity. Hence, it is sufficient to consider the fluctuations of the dynamic response. The static response to the mean of the dynamic excitation can be ignored.

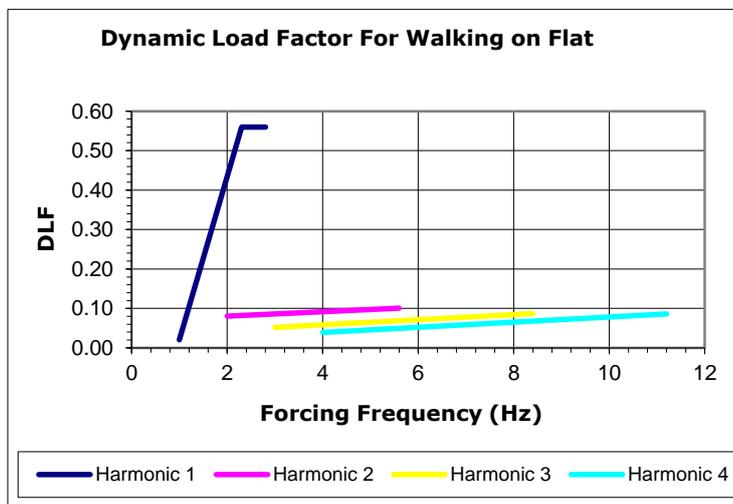
Typical single and continuous footfall force time histories are presented. These are inharmonic periodic signals.



The first graph shows the Load/(Static Weight) to be greater or less than 1.0, which represents the static weight. Taking the mean as the static weight, the dynamic load excitation can then be defined with positive and negative fluctuations as shown on the second graph. It can be seen that the ‘positive’ downward force phase is generally shorter and of higher magnitude than the negative force phase. Note that the positive and negative here are relative to the static weight of the person. These time histories can be decomposed into their frequency components by Fourier Series analysis. The amplitude of the frequency component time histories relative to the static load is called the dynamic load factor, clearly less than 1.0 because these are the amplitude of the fluctuations about the mean static weight. The dynamic load factor (not the (magnitude of the) dynamic amplification factor) multiplied by the static load is hence the amplitude of the dynamic excitation, for a particular harmonic frequency component.



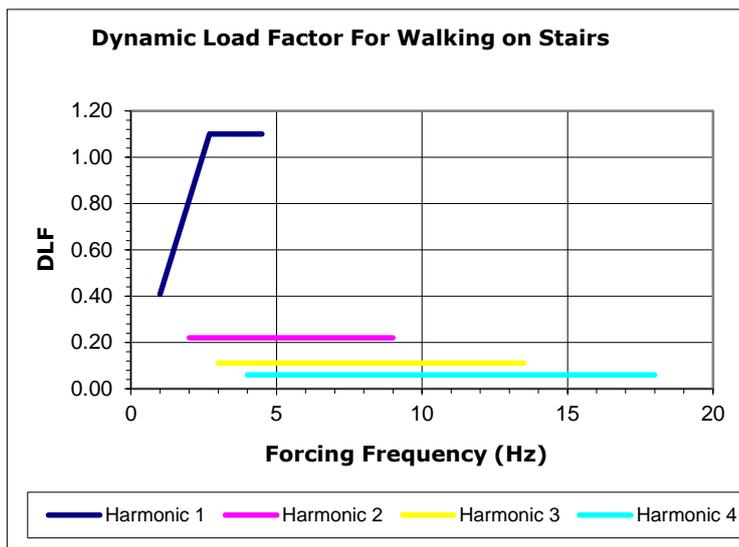
Bear in mind that the total response will then be that of the dynamic response plus the mean of the dynamic excitations, i.e. the static weight of the person, plus the static response due to the static loads on the structure. As always done in frequency domain analyses, the mean is applied separately to the dynamic fluctuations. However, in the case of floor excitations, we are usually only interested in the dynamic acceleration or velocity response and not the total (static + dynamic) displacement response. These curves show that the footfall time histories can be decomposed into **4 harmonic components**. The Arup DLF curves for walking on flat (floors and footbridges) are reproduced for clarity.



The DLFs are the magnitudes of the sinusoidal harmonic components (for frequencies at 1, 2, 3 ... etc ... of the walking frequency) derived by Fourier Analysis, that when added together, will recreate the original force time history.

Harmonic	Excitation Frequency range (Hz)	Mean value	Coefficient of variation	Design Value
1	1 – 2.8	$0.37(f - 0.95), \geq 0.5$	0.17	$0.41(f - 0.95), \geq 0.56$
2	2 - 5.6	$0.054 + 0.0044f$	0.40	$0.069 + 0.0056f$
3	3 - 8.4	$0.026 + 0.0050f$	0.40	$0.033 + 0.0064f$
4	4 - 11.2	$0.010 + 0.0051f$	0.40	$0.013 + 0.0065f$

The Arup DLF curves for walking on stairs are also presented.



Note that the DLFs for walking on stairs are greater than for walking on flats. Also, the frequency range is higher, i.e. first harmonic frequency range is up to 4.5Hz for walking on stairs compared to 2.8Hz for walking in flats.

Multi-Modal Response Analysis

It is the practice to perform resonant response calculations for single person walking excitations, albeit at the most critical footfall rate, not necessarily the highest or the lowest footfall rate, but *usually* that which corresponds to the natural frequency of the structure. We say *usually* because the DLF varies. For instance, if we had a stair of fundamental frequency 2.0Hz, we need to consider the response due to excitation at 2.0Hz to maximize the dynamic amplification (but at a lower DLF) and also the response at 2.7Hz to consider the maximum DLF, but off-resonant response, but yet as close to 2.0Hz with maximum DLF. If several people are walking together, unless they are marching in a disciplined manner, the footfall rates and phasing will not be coordinated, and hence the response will not increase in proportion to the number of people.

The first harmonic component varies from frequencies of 1Hz to 2.8Hz for walking on flats and from 1Hz to 4.5Hz for walking on stairs. The DLF versus harmonic excitation frequency curves are very illustrative. Since resonant vibrations response is far more significant than impulsive response in any field of dynamic excitation, we would ideally like to avoid any form of resonance. For footfall vibrations, resonance with the first harmonic is clearly most damaging since the DLF associated with the first harmonic is the largest. Hence if the fundamental vertical natural frequency of the structure is significantly more than 2.8Hz or 4.5Hz (whichever relevant), the resonance with the first harmonic is avoided. Similarly, if fundamental vertical natural frequency of the structure is significantly more than 5.6Hz or 9.0Hz (whichever relevant), the resonance with the second harmonic is avoided. Finally, if fundamental vertical natural frequency of the structure is significantly more than 8.4Hz or 13.5Hz (whichever relevant), the resonance with the third harmonic is avoided, although to obtain resonance with the 3rd (or 4th for that matter) harmonic is quite difficult anyway. It should be noted that **FOR STAIRS** resonance of the

second harmonic has been observed but **NOT** of the third and higher harmonics, probably because of imprecise loading. If we were to consider resonance of modes due to 4th harmonic excitations, we would need to consider modes up to 11.2Hz and 18.0Hz (for walking on flat and stairs respectively). **Let us then solve for modes up to say 15Hz and 22Hz (flat and stair respectively) to also capture the off-resonant responses of modes higher than 11.2Hz and 18.0Hz.**

The calculation procedure is now illustrated. A conventional modal frequency domain analysis is performed. The equation of motion in the time domain is

$$[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + [K]\{u(t)\} = \{P(t)\}$$

In the frequency domain we let

$$\begin{aligned} \{u(t)\} &= \text{Re al} \left[[\Phi] \{\xi(\omega)\} e^{i\omega t} \right] \text{ where } \{\xi(\omega)\} \text{ is a complex modal displacement response function vector} \\ \{P(t)\} &= \text{Re al} \left[\{P(\omega)\} e^{i\omega t} \right] \text{ where } \{P(\omega)\} \text{ is a harmonic loading function vector} \end{aligned}$$

And orthogonalizing the coupled equations, leads to a set of uncoupled equations in the form for a mode i

$$-\omega^2 M_i \xi_i(\omega) + i C_i \omega \xi_i(\omega) + K_i \xi_i(\omega) = P_i(\omega)$$

Rearranging for the complex modal response (in modal space)

$$\begin{aligned} \xi_i(\omega) &= \frac{P_{0i}}{-\omega^2 M_i + i C_i \omega + K_i} \\ &= \frac{P_{0i}}{-\omega^2 M_i + K_i + i C_i \omega} \\ &= \frac{P_{0i} / K_i}{\left(1 - \omega^2 / \omega_{ni}^2\right) + i(2\zeta_i \omega / \omega_{ni})} \end{aligned}$$

Instead of the polar form, this can be written in terms of the real and imaginary parts

$$\begin{aligned} \xi_i(\omega) &= \frac{P_{0i}}{K_i} \left[\frac{\left(1 - \omega^2 / \omega_{ni}^2\right)}{\left(1 - \omega^2 / \omega_{ni}^2\right)^2 + \left(2\zeta_i \omega / \omega_{ni}\right)^2} - i \frac{\left(2\zeta_i \omega / \omega_{ni}\right)}{\left(1 - \omega^2 / \omega_{ni}^2\right)^2 + \left(2\zeta_i \omega / \omega_{ni}\right)^2} \right] \\ \xi_i(\omega) &= \xi_{i\text{REAL}}(\omega) - i \xi_{i\text{IMAG}}(\omega) \end{aligned}$$

Thus, the steady-state solution in the time domain

$$\begin{aligned} \{u(t)\} &= \text{Re al} \left[[\Phi] \{\xi_{\text{REAL}}(\omega) - i \xi_{\text{IMAG}}(\omega)\} e^{i\omega t} \right] \\ \{u(t)\} &= \text{Re al} \left[[\Phi] \{\xi_{\text{REAL}}(\omega) - i \xi_{\text{IMAG}}(\omega)\} (\cos \omega t + i \sin \omega t) \right] \\ \{u(t)\} &= [\Phi] \left\{ \{\xi_{\text{REAL}}(\omega)\} \cos \omega t + \{\xi_{\text{IMAG}}(\omega)\} \sin \omega t \right\} \end{aligned}$$

Note that this expression is to be interpreted as

$$\{u(t)\} = \left[\begin{array}{cccc} \{\phi\}_1 & \dots & \{\phi\}_i & \dots & \{\phi\}_n \end{array} \right] \left\{ \begin{array}{l} \frac{P_{01}}{K_1} (D_{1\text{REAL}}(\omega) \cos \omega t + D_{1\text{IMAG}}(\omega) \sin \omega t) \\ \dots \\ \frac{P_{0i}}{K_i} (D_{i\text{REAL}}(\omega) \cos \omega t + D_{i\text{IMAG}}(\omega) \sin \omega t) \\ \dots \\ \frac{P_{0n}}{K_n} (D_{n\text{REAL}}(\omega) \cos \omega t + D_{n\text{IMAG}}(\omega) \sin \omega t) \end{array} \right\}$$

where

$$D_{i\text{REAL}}(\omega) = \frac{(1 - \omega^2 / \omega_{ni}^2)}{(1 - \omega^2 / \omega_{ni}^2)^2 + (2\zeta_i \omega / \omega_{ni})^2}$$

$$D_{i\text{IMAG}}(\omega) = \frac{(2\zeta_i \omega / \omega_{ni})}{(1 - \omega^2 / \omega_{ni}^2)^2 + (2\zeta_i \omega / \omega_{ni})^2}$$

Since $\text{MAX}(A\cos\omega t + B\sin\omega t) = (A^2 + B^2)^{1/2}$, at a particular DOF j , the steady state displacement response will be

$$\text{Steady - State Response, } u_{j\text{MAX}} = \sqrt{\left(\sum_i^n \frac{\rho_{0i}}{K_i} \phi_{ij} D_{i\text{REAL}}(\omega) \right)^2 + \left(\sum_i^n \frac{\rho_{0i}}{K_i} \phi_{ij} D_{i\text{IMAG}}(\omega) \right)^2}$$

The expression for acceleration can be obtained simply by differentiating the displacement expression twice w.r.t. time. Hence there will be a multiple of the square of the excitation frequency, ω^2 .

The above expression is exact for excitations at a certain point long enough for steady-state conditions to be achieved. In practice of course, the number of footfalls are **limited (a pedestrian is only effective in generating resonant response when walking along a single half sine wave of the mode shape; as he moves on to the next, out of phase half sine wave he begins to cancel out the vibration he has just generated) and not all these footfalls are applied at the same mode-shape point (many of the steps will have been towards the edge or the less active part of the half sine and would have been less effective in generating response)**. Hence, only a proportion of the maximum possible steady-state is the achieved. The maximum proportion of steady-state response for crossing a sinusoidal mode shape can be estimated as

$$r = 1 - e^{-2\pi N} \quad N \approx 0.55n \frac{L}{l}$$

where N is approximately 0.55 of the total number of vibration cycles occurring as the span is crossed. The total number of cycles for first harmonic loading is the span distance L divided by the stride length l , and for higher harmonics this number of cycles is multiplied by the harmonic number n . **This expression for r is best applied ONLY to the imaginary parts in this procedure.**

It is also noted that the harmonic components of the footfall forcing function are not in phase. The actual phases of these harmonic components vary considerably between different persons and walking speeds. There is therefore no general relationship of phases that can be used to generate a unique time history. Hence it is recommended that the maximum response amplitude be estimated by means of the SRSS combination of the individual responses of each harmonic.

The peak acceleration can be compared to peak acceleration acceptability criteria in the 1/3 octave bands. Alternatively, the RMS acceleration response is derived from the maximum response (by dividing by $\sqrt{2}$ since simple harmonic motion) and comparison to the RMS acceptability criteria in the 1/3 octave bands is made.

Let us illustrate the resonant response calculation for ALL modes exactly for the response at ONE node due to excitation at ONE harmonic frequency, say 5.14Hz (corresponding to the second harmonic of a walking frequency of 2.57Hz). This harmonic frequency corresponds to the first mode. Hence the first mode resonates whilst the rest of the modes present off-resonant response.

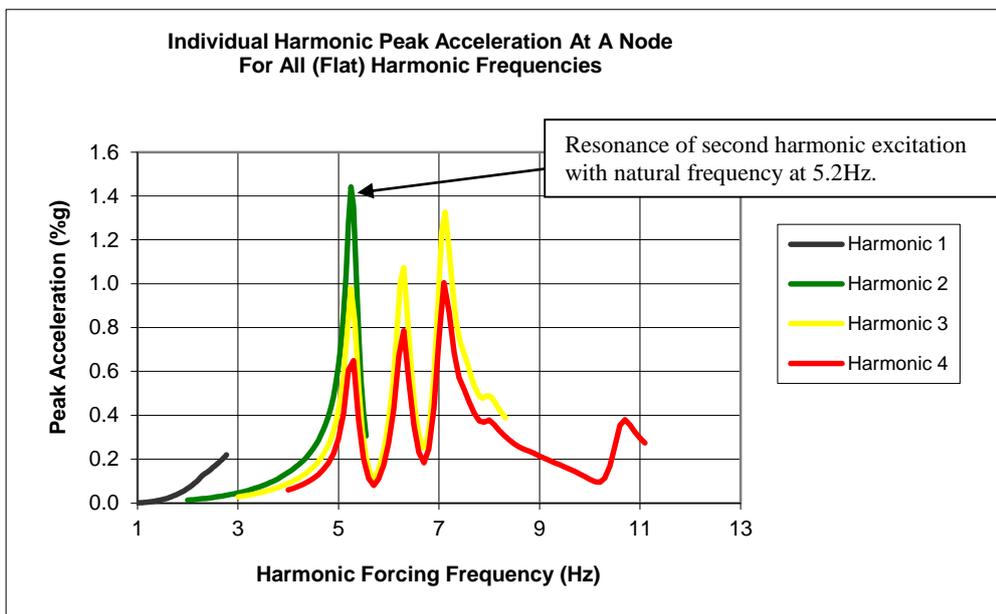
Mode	Natural Frequency, f_n (Hz)	Excitation Frequency, f (Hz)	f/f_n	Modal Damping	$D_{i\text{REAL}}$	$D_{i\text{IMAG}}$
1	5.14	5.14	1.00	0.03	0.000	16.667
2	5.39	5.14	0.95	0.03	7.890	4.982
3	6.30	5.14	0.82	0.03	2.928	0.429
4	8.28	5.14	0.62	0.03	1.621	0.098

Say that the excitation node and response node is the same and corresponds to unity modal displacement for mode 1. The amplitude of the modal force for mode 1 is thus, $\rho_{01} = 1.0 \times \text{DLF} \times 700\text{N} = 1.0 \times 0.098 \times 700 = 68.5\text{N}$.

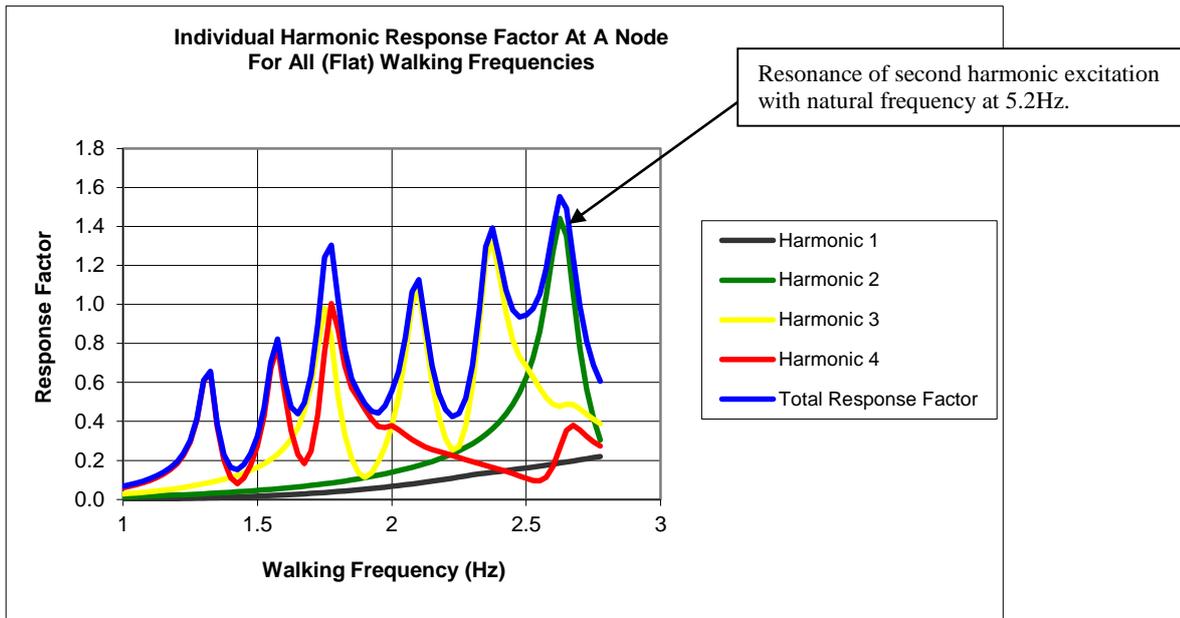
Modal Mass	Modal Displacement at Excitation Node, ϕ_e	Modal Force Amplitude, $\rho_{0i} = 68.5\phi_e$	Modal Displacement at Response Node, ϕ_r	$\phi_r(\rho_{0i}/M) \cdot (f/f_n)^2 \cdot D_{\text{IREAL}}$	$\phi_r(\rho_{0i}/M) \cdot (f/f_n)^2 \cdot D_{\text{IMAG}}$
29551	1.000	68.500	1.000	0.00000	0.03863
29551	0.953	65.260	0.953	0.01510	0.00953
29551	0.816	55.899	0.816	0.00301	0.00044
29551	0.621	42.552	0.621	0.00056	0.00003
				0.01866	0.04864

The steady-state response is then $(0.01866^2 + 0.04864^2)^{1/2} = 0.0521\text{ms}^{-2}$. Assuming a stride length of 0.75m the number of steps to cross the span is 16, resulting in 32 cycles of vibration at the 2nd harmonic frequency. The proportion of steady state response for 3% damping is then obtained, $r = 1 - e^{-2\pi \cdot 0.03N} = 0.96$ where $N = 0.55 \times 32 = 17.6$. Hence the corrected steady-state response is $0.96 \times 0.0521 = 0.050\text{ms}^{-2}$. And thus the response factor, $R = 0.050/0.007 = 7.1$.

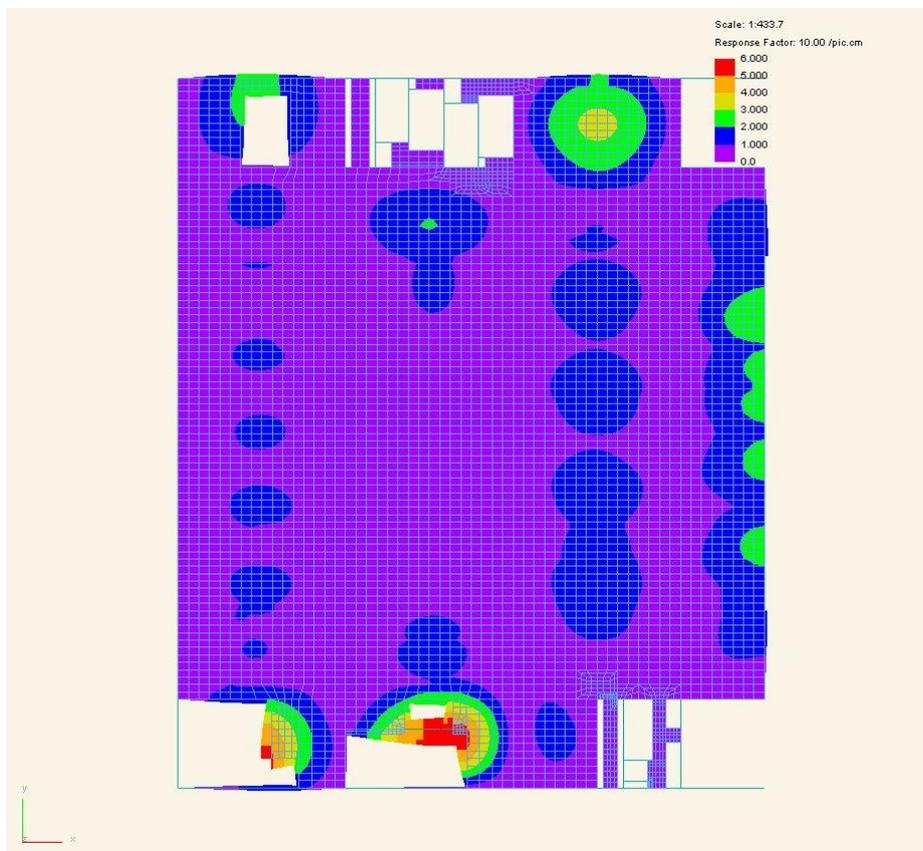
The peak acceleration responses at A NODE due to excitation at A NODE (not necessarily but usually the same node), for ALL harmonic forcing frequencies (from walking on flat) can be presented as follows. The graph shows that the maximum response occurs due to the second harmonic at 5.2Hz. There is clearly a significant (to the response at THIS NODE) natural mode at this frequency. There are also natural modes at 6.3Hz and 7.1Hz, but the maximum response comes from the second harmonic, which has a larger DLF. Notice that the first harmonic, which has the greatest DLF, does not peak because there is no significant natural mode within the frequency range of the first harmonic.



The response at A NODE due to excitation at A NODE can also be presented in terms of the walking frequency i.e. the frequency of the first harmonic. The former graph is more illustrative as it shows the response in terms of the individual harmonic frequencies, not just the first harmonic (i.e. walking frequency).



The MAXIMUM response at EACH AND EVERY NODE giving consideration to ALL the harmonics altogether can be computed and presented as follows.



Fundamental Mode Response Analysis – Hand Calculation

A single fundamental mode response usually provides good approximations to the *maximum* response because of the fact that different modes are out of phase with each other, their maximums at a point not occurring at the same time. Hence, for a floor plate, the *maximum* response in the middle of the plate is usually governed just by the fundamental mode. Higher modes have higher natural frequencies and hence the modal stiffness will be larger, resulting in a lower modal response. This approximation begins to fall apart when there are modes closely spaced, in which case the multi-modal approach should be used. The multi-modal approach should also be used if the response due to the first harmonic exciting a mode and higher harmonics exciting another mode is required. Of course, the multi-modal approach also considers multiple modes (albeit most of them contributing off-resonant response).

To ascertain the most significant mode for the response to excitation at a particular node, scroll (visually or numerically) through all modes and pick out the modes with the greatest displacement value at that node. This gives the modes with the highest modal force. Of these, mode with the lowest modal mass will obviously yield the highest response. Hence, pick out the mode with the lowest modal mass (by FE computation or otherwise) or by visual inspection knowing that modes that mobilize more of the structure will be have a larger modal mass associated with it.

Because we need not consider the phase information (since no mode combinations are required), we do not need to express the dynamic amplification in terms of its real and imaginary parts. Instead, we consider the magnitude of the dynamic amplification function straight away. This gives the **steady-state** (i.e. **peak**) **acceleration** response in physical coordinates as

$$a_{\text{peak}} = \phi_{\text{response}} \phi_{\text{excitation}} \omega^2 \frac{p_0}{\omega_{ni}^2 M_i} \frac{1}{\sqrt{\left(1 - \omega^2 / \omega_{ni}^2\right)^2 + \left(2\zeta_i \omega / \omega_{ni}\right)^2}}$$

And at **resonance** ($\omega = \omega_{ni}$),

$$a_{\text{peak}} = \phi_{\text{response}} \phi_{\text{excitation}} \frac{p_0}{M_i} \frac{1}{2\zeta_i}$$

Inculcating the proportion of steady-state response achievable for crossing span

$$a_{\text{peak}} = \phi_{\text{response}} \phi_{\text{excitation}} r \frac{p_0}{M_i} \frac{1}{2\zeta_i} \quad \text{where } r = 1 - e^{-2\pi\zeta N}, N = 0.55n \frac{L}{\ell}$$

For the maximum (peak) response with excitation at the maximum point of the mode shape and the weight of a person as 70kg (about 700N),

$$a_{\text{peak}} = (1.0)(1.0)r \frac{700 \times \text{DLF}}{M_i} \frac{1}{2\zeta_i} \quad \text{where } r = 1 - e^{-2\pi\zeta N}, N = 0.55n \frac{L}{\ell}$$

where the DLF corresponds to the excitation frequency equal (as resonance response intended) to the natural frequency of the mode under consideration.

The peak acceleration can be compared to peak acceleration acceptability criteria in the 1/3 octave bands. Alternatively, the RMS acceleration response is derived from the maximum response (by dividing by $\sqrt{2}$ since simple harmonic motion) and comparison to the RMS acceptability criteria in the 1/3 octave bands is made. Hence, if the excitation is between $4\text{Hz} < f_{\text{harmonic excitation}} < 8\text{Hz}$, then

$$\text{Response Factor} = \frac{a_{\text{peak}}}{0.007} \quad \text{or} \quad \text{Response Factor} = \frac{a_{\text{RMS}}}{0.005} = \frac{a_{\text{peak}} / \sqrt{2}}{0.005} = \frac{a_{\text{peak}}}{0.007}$$

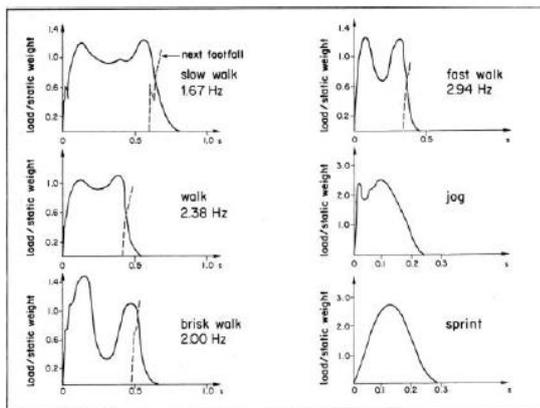
1.1.3.1.2 Transient Excitations For Modes Above 15Hz (Flat) and 22Hz (Stairs)

In this **LINEAR TIME DOMAIN** solution, the static response must be added to the dynamic response if the dynamic analysis is performed about the initial undeflected (by the static loads) state with only the dynamic loads applied, hence causing the dynamic response to be measured relative to the static equilibrium position. **Hence, the total response = the dynamic response + the static response to static loads.**

Alternatively, in this LINEAR TIME DOMAIN solution, if the dynamic analysis is performed with the deflected static shape as initial input and the static loads maintained throughout the dynamic excitations, the total or absolute response (static and dynamic) is obtained straight away from the dynamic analysis. **Hence total response = dynamic response (which already includes the static response to static loads).**

However, usually in footfall induced response, it is not the total response which is of importance, as the criteria is not from a structural safety viewpoint, but rather from a human comfort viewpoint, the latter of which is measured in terms of acceleration and velocity. Hence, it is sufficient to consider dynamic response due to the impulse excitations. This ignores the static response to static loads such as that of the floor, but does inherently include the static response to the mean of the dynamic excitation (unlike the frequency domain approach).

Typical single footfall force time histories are presented.



Multi-Modal Response Analysis

Where resonance excitations cannot occur, the response will be governed by an impulsive type response, i.e. the starting transient response will be more significant than the steady-state response (or that during the build up of the steady-state response), giving a much lower dependency of response on the exact frequency and on damping.

The calculation procedure is illustrated. The dynamic equation of motion in the time domain is

$$[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + [K]\{u(t)\} = \{P(t)\}$$

Orthogonalizing the coupled equations, leads to a set of uncoupled equations in the form for a mode i

$$M_i \ddot{\xi}_i(t) + C_i \dot{\xi}_i(t) + K_i \xi_i(t) = P_i(t) \quad \text{for the } i^{\text{th}} \text{ mode}$$

On rearranging, the modal response in modal coordinates ξ_i

$$\xi_i(t) = D_i(t) \frac{P_{0i}}{K_i} = D_i(t) \frac{P_{0i}}{\omega_{ni}^2 M_i}$$

Finally, the response in physical coordinates $u_i(t)$ are

$$\{u(t)\} = [\Phi]\{\xi(t)\}$$

In the case where the excitation is impulsive ($t_i/T < \sim 0.2$), the maximum modal displacement response will be

$$\xi_{i\max} = \frac{1}{M_i \omega_{ni}} \int_{\tau=0}^{\tau=t} \phi_i p(\tau) d\tau \quad \text{undamped}$$

$$\xi_{i\max} = \frac{1}{M_i \omega_{di}} \int_{\tau=0}^{\tau=t} \phi_i p(\tau) d\tau \quad \text{damped}$$

Response formula becomes less valid for higher modes of vibration as $t_i/T_i > \sim 0.2$

where ϕ_i = modal component at excitation point

And, the displacement response in physical coordinates $u_i(t)$ are

$$\{u(t)\} = [\Phi]\{\xi(t)\}$$

In time domain analyses the maxima of the modal responses in physical space do not occur at the same time. Hence, the exact procedure would be to write the total response in physical space as above and only then maximize the function. If however the modal responses were to be maximized, then some sort of combination technique such as the SRSS or CQC should be employed.

For impulsive excitation though, the maximum response occurs at the first cycle of ensuing vibration after impact. The phase difference between modes is quite insignificant as it takes time for damping to affect the phase between the modes. Hence the maximum response can be approximated as a simple summation of the maximum response of the individual modes.

For response due to impulsive type excitations, only one footfall excitation is required as the maximum occurs at the very first cycle. The ensuing vibration is the unforced vibration of the structure at its damped natural frequencies. Hence the maximum (peak) modal velocity response is obtained (by differentiating w.r.t time)

$$\dot{\xi}_{i\max} = \frac{1}{M_i} \int_{\tau=0}^{\tau=t} \phi_i p(\tau) d\tau$$

For flats, the value of the (empirical) effective impulse, $I_{\text{effective}}$ (with 25% exceedance probability) can be taken as

$$I_{\text{effective}} = \int_{\tau=0}^{\tau=t} \phi_i p(\tau) d\tau = \phi_i 54 \frac{f^{1.43}}{f_n^{1.30}} \text{Ns}$$

where f is the excitation frequency and f_n the natural frequency.

For stairs, the value of the (empirical) effective impulse, $I_{\text{effective}}$ can be taken as

$$I_{\text{effective}} = \int_{\tau=0}^{\tau=t} \phi_i p(\tau) d\tau = \phi_i \frac{150}{f_n} \text{Ns}$$

And, the velocity response in physical coordinates are

$$\{\dot{u}(t)\} = [\Phi]\{\dot{\xi}(t)\}$$

The maximum (peak) velocity response is approximated by the simple addition of the maximum individual modal velocity responses, although this is not strictly correct. With modal methods, the quantity for every mode (be it the displacement, velocity, acceleration, stress or force) must be combined before being maximized. Note that you can derive the velocity and acceleration for every mode to be combined, and not just the displacement. But the fact remains that the maximizing must be performed after modal combination. But because we are assuming that all modes are responding in an impulsive nature (although clearly this assumption is NOT TRUE for higher modes of vibration) we shall maximize the response of individual modes before combination, because we are assuming that the phase difference between modes to be insignificant for the reason outlined above (that all modes response in an impulsive nature) and that damping is not prominent to affect the phase in the first cycle.

Alternatively, the correct method of modal combination should be used. Here the different modal responses in the physical coordinates are combined before maximizing the function, which is as follows.

$$\dot{u}(t) = \sum_{i=1}^N \phi_i \dot{\xi}_i e^{-\zeta_i \omega_{ni} t} \sin \omega_{di} t$$

The peak velocity CANNOT be compared to peak velocity acceptability criteria because the response is characterized by a series of decaying transient vibrations and hence the peak velocity is unrepresentative in perceptibility terms. An effective peak value of $\sqrt{2}$ times the RMQ velocity response in the 1/3 octave bands can be used in comparison to the peak velocity acceptability in the 1/3 octave bands. Alternatively, the RMS or RMQ velocity response in the 1/3 octave bands can be compared to the RMS or RMQ velocity acceptability criteria in the 1/3 octave bands.

Fundamental Mode Response Analysis – Hand Calculation

For a fundamental mode response analyses, the maximum (peak) velocity response is given by

$$v_{\text{peak}} = \phi_{\text{response}} \frac{I_{\text{effective}}}{M_i}$$

For flats, the value of the (empirical) effective impulse, $I_{\text{effective}}$ (with 25% exceedance probability) can be taken as

$$I_{\text{effective}} = \int_{\tau=0}^{\tau=t} \phi_i p(\tau) d\tau = \phi_{\text{excitation}} 54 \frac{f^{1.43}}{f_n^{1.30}} \text{Ns}$$

where f is the excitation frequency and f_n the natural frequency.

For stairs, the value of the (empirical) effective impulse, $I_{\text{effective}}$ can be taken as

$$I_{\text{effective}} = \int_{\tau=0}^{\tau=t} \phi_i p(\tau) d\tau = \phi_{\text{excitation}} \frac{150}{f_n} \text{Ns}$$

The peak velocity CANNOT be compared to peak velocity acceptability criteria because the response is characterized by a series of decaying transient vibrations and hence the peak velocity is unrepresentative in perceptibility terms. An effective peak value of $\sqrt{2}$ times the RMQ velocity response in the 1/3 octave bands can be used in comparison to the peak velocity acceptability in the 1/3 octave bands. Alternatively, the RMS or RMQ velocity response in the 1/3 octave bands can be compared to the RMS or RMQ velocity acceptability criteria in the 1/3 octave bands. The approximate equivalent RMS velocity response is

$$v_{\text{RMS}} = \frac{2}{3} \frac{v_{\text{peak}}}{\sqrt{2}}$$

Hence, if the excitation is $f_{\text{harmonic excitation}} > 8\text{Hz}$, then

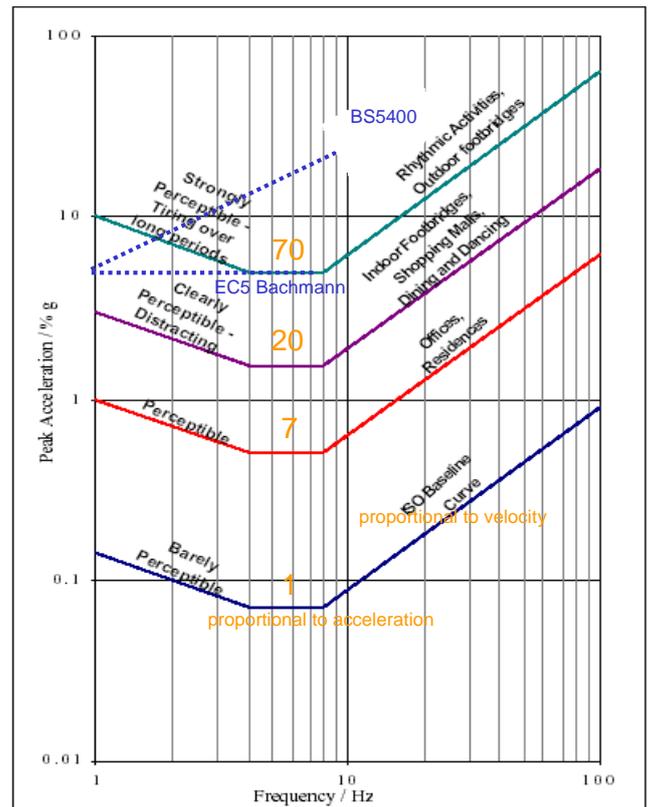
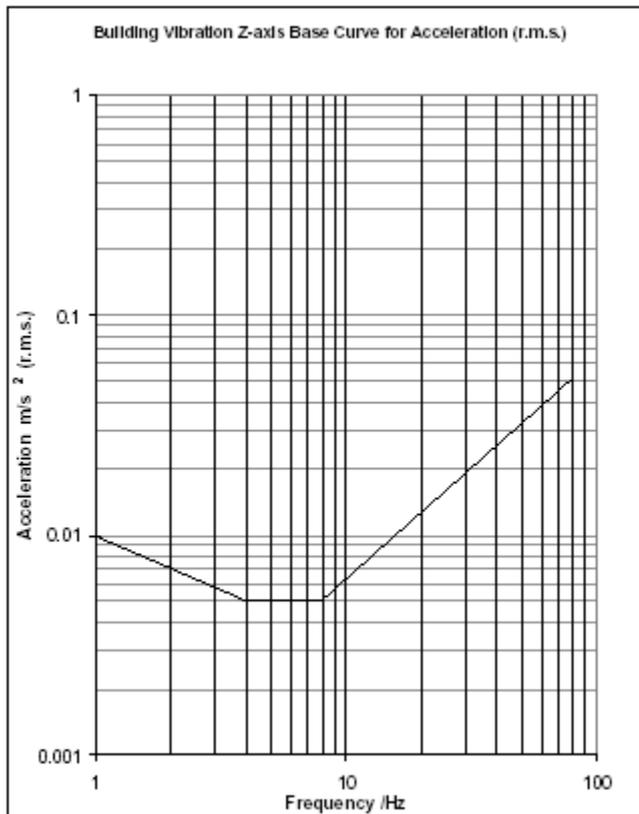
$$\text{Re sponse Factor} = \frac{v_{\text{RMS}}}{0.0001} \quad \text{or} \quad \text{Re sponse Factor} = \frac{a_{\text{RMS}}}{\sim 0.005} = \frac{2\pi f v_{\text{RMS}}}{\sim 0.005}$$

Expanding the former expression,

$$\text{Re sponse Factor} = \frac{v_{\text{RMS}}}{0.0001} = \frac{1}{0.0001} \frac{2}{3} \frac{v_{\text{peak}}}{\sqrt{2}} \approx 4700 v_{\text{peak}}$$

1.1.3.2 Serviceability Acceptance Criteria

The perceptibility and then acceptance criteria can be established in terms of peak, RMS or RMQ of a signal. RMQ has been shown to be the most reliable parameter. BS6472:1992 states the threshold of human perception in terms of the baseline RMS vertical acceleration versus vibration frequency.



Between 4 and 8Hz, the baseline RMS acceleration is 0.005ms^{-2} corresponding to a baseline peak acceleration (for steady state signals) of 0.007ms^{-2} . The rising acceleration criteria for frequencies above 8Hz correspond to a constant RMS velocity of 0.0001ms^{-1} . The so-called Response Factor is the number of multiples the response acceleration or velocity is that of this baseline.

Structure	Acceptable Response Factor
Hospital Floor	1
Special Office Floor	4
Normal Office Floor	8
Busy Office Floor	12
Heavy Public Stairs	24
Light Office Stairs	32
Very Light Escape Stairs	64
External Footbridge	50+

Increasingly over the last thirty years vibration-sensitive equipment has been used within research laboratories and production facilities in the fields of microelectronics, opto-electronics, metrology, biotechnology and medicine. Vibration levels far below human perception thresholds are normally required for these facilities, because vibration at the equipment support points can cause internal components, study specimens, or items being produced, to move relative to each other. Therefore, these facilities call for close attention to vibrations, particularly on suspended floor structures. For example, a typical semiconductor production facility might have for its $5,000\text{m}^2$ clean room a 1m deep concrete grillage floor supported on columns spaced at 4m x 4m. A medical laboratory floor, which might

have vibration requirements an order of magnitude less stringent, might be designed with conventional slab/deck/steel-framing schemes similar to, but significantly stiffer and heavier than conventional office building framing. General vibration criteria as measured in one-third octave bands of frequency over the frequency range 8 to 100 Hz for laboratories are as follows.

Criterion curve	Max. velocity level* µm/sec (RMS)	Detail size** microns	Description of use
Workshop (ISO2631 and BS6472) R = 8, ASHRAE J	800	N/A	Distinctly perceptible vibration. Appropriate to workshops and non-sensitive areas.
Office (ISO2631 and BS6472) R = 4, ASHRAE I	400	N/A	Perceptible vibration. Appropriate to offices and non-sensitive areas.
Residential day (ISO2631 and BS6472) R = 2, ASHRAE H	200	75	Barely perceptible vibration. Appropriate to sleep areas in most instances. Probably adequate for computer equipment, probe test equipment and low-power (to 20X) microscopes.
Operating theatre (ISO2631 and BS6472) R = 1, ASHRAE F	100	25	Threshold of perception. Suitable for sensitive sleep areas. Suitable in most instances for microscopes to 100X and for other equipment of low sensitivity.
VC-A (BBN-A or ASHRAE E) R = 0.5	50	8	Adequate in most instances for optical microscopes to 100X, microbalances, optical balances, proximity and projection aligners, etc.
VC-B (BBN-B or ASHRAE D) R = 0.25	25	3	An appropriate standard for optical microscopes to 1000X, inspection and lithography equipment (including steppers) to 3 micron line widths.
VC-C (BBN-C or ASHRAE C) R = 0.125	12.5	1	A good standard for most lithography and inspection equipment to 1-micron detail size.
VC-D (BBN-D or ASHRAE B) R = 0.0625	6	0.3	Suitable in most instances for the most demanding equipment including electron microscopes (TEMs and SEMs) and E-Beam systems, operating to the limits of their capability.
VC-E (BBN-E or ASHRAE A) R = 0.03125	3	0.1	A difficult criterion to achieve in most instances. Assumed to be adequate for the most demanding of sensitive systems including long path, laser-based, small target systems and other systems requiring extraordinary dynamic stability.
Notes			
* As measured in one-third octave bands of frequency over the frequency range 8 to 100 Hz.			
** The detail size refers to the line widths for microelectronics fabrication, the particle(cell) size for medical and pharmaceutical research, etc. the values given take into account the observation that the vibration requirements of many items depend upon the detail size of the process.			

1.1.3.3 Practical Solutions

Flats or stairs that have fundamental frequency falling near that of the first harmonic are completely unsuitable for service.

With stairs, if the stiffness of the stringers can be increased by having deeper beam sections or by tying the stringers together, then the natural frequencies of the stair would increase, hence making it less susceptible to human induced footfall vibrations. Another solution would be to have greater mass by incorporating heavier treads or by using glass along the hand railings, hence reducing the modal response, although of course the natural frequencies may decrease too much as to make it again susceptible to footfall excitation frequencies. Alterations to the boundary conditions can also be recommended at times to increase the natural frequencies of the stair, for instance by making the extremities of the stair encastre or if possible by breaking the span with a column, a beam or a hanger. As a last resort, a TMD would have to be designed and installed if all other solutions prove unacceptable to the architects.

On floor vibration problems, the uniformity of the floor between adjacent bays will help lower the response. This is because the more uniform the bays are, the greater the contribution of the mass from adjacent bays into the modal mass of a particular mode. Of course, the higher the modal mass of a particular mode, the smaller will be the response. To illustrate this, consider two adjacent bays, one much stiffer than the other, here by virtue of the shorter length. Clearly, because of the small components of the eigenvector in the stiff bay, its contribution to the modal mass will be minimal. On the other hand, if the bays were of similar stiffness, the eigenvector will be more uniform and there will be a greater contribution to the modal mass from both bays.



Another case is bays that are adjacent to holes or isolated bays. The modal mass is provided by the mass only in that bay. No contribution from adjacent bays. The modal mass is thus low and the mode becomes a local mode. This is akin to a bridge really. Thus, the response can be high.

Stiffness, mass and damping are the three parameters that the designer can vary to improve the dynamic performance of a floor when the basic geometry has been fixed. The true implications of making changes can only be assessed if there is proper understanding that it is a dynamic problem under consideration, with acceleration (not deflection) usually the quantity of concern. The way in which footfall force vary with frequency must also be properly appreciated. Whilst there is a decrease in footfall forces as the harmonic number increases, there is also an increase in footfall force with increasing walking speed within each harmonic, especially for the first harmonic. It is not therefore always advantageous to increase the natural frequency of a floor to reduce accelerations. Unless increasing the natural frequency takes it out of the range of susceptibility to a particular harmonic, the increase in frequency may be counter-productive. There is a strong case for saying that a floor of 3.0 to 3.6Hz natural frequency has a lower likelihood of excessive vibration than one of 3.6 to 4.4Hz. Both ranges are susceptible to the 2nd harmonic of footfall forces, but the forces are lower for the lower walking speeds that would excite the lower frequency range.

Increasing the damping is always beneficial, but cannot currently be done inexpensively. Increasing the mass is also always beneficial, unless by so doing the natural frequency is reduced below a threshold that makes the floor susceptible to a lower harmonic of footfall force. Increasing the stiffness is not always beneficial; as noted above, increasing the natural frequency can render a floor susceptible to higher footfall forces from faster walking rates. Additional stiffness which takes the natural frequency above one of the thresholds is however beneficial.

Experience has shown that if responses as high as $R=8$ are measured in offices there is occasional adverse comment, particularly if these levels are reached regularly, and lower targets ($R=4-6$) are recommended as necessary to remove the likelihood of such comment.

1.1.4 (Non-Crowd or Asynchronous) Human Induced Vertical Footfall Vibrations from Sports in Halls ²

In this **LINEAR FREQUENCY DOMAIN solution**, not only that the static response has to be added separately, but also the mean of the dynamic excitation has also got to be added separately as a static response. This is because the mean of the dynamically applied force is not included in the dynamic excitations. **Hence the total response in this frequency domain dynamic analysis = static response to mean of dynamic excitation + dynamic response + static response to static loads.** However, usually in footfall induced response, it is not the total response which is of importance, as the criteria is not from a structural safety viewpoint, but rather from a human comfort viewpoint, the latter of which is measured in terms of acceleration and velocity. Hence, it is sufficient to consider the fluctuations of the dynamic response.

The established dynamic response calculation theories for walking on flats and stairs are generally equally applicable to predicting (single person) induced vibrations due to sports in halls. Certain terms such as the maximum proportion of steady-state response for crossing a sinusoidal mode shape

$$r = 1 - e^{-2\pi\zeta N}$$

where N is approximately 0.55 of the total number of vibration cycles occurring as the span is crossed may or may not be applicable as the excitation may be assumed located at the same spot, as the critical consideration.

The DLFs for the different activities are presented as follows according to Bachmann.

Activity	Activity Rate (Hz)	1 st Harmonic DLF	2 nd Harmonic DLF	3 rd Harmonic DLF
Running	2.0Hz to 3.0Hz	1.6	0.7	0.2
Jumping	Normal 2.0Hz	1.8	1.3	0.7
	Normal 3.0Hz	1.7	1.1	0.5
	High 2.0Hz	1.9	1.6	1.1
	High 3.0Hz	1.8	1.3	0.8
	Standing 0.6Hz	DLF _{1/2} = 0.5		

For a single mode, the maximum (peak) resonant response with excitation at the maximum point of the mode shape and the weight of a person as 70kg (about 700N), the peak acceleration is

$$a_{\text{peak}} = (1.0)(1.0) \frac{700 \times \text{DLF}}{M_i} \frac{1}{2\zeta_i}$$

where the DLF corresponds to the excitation frequency equal (as resonance response intended) to the natural frequency of the mode under consideration.

If the excitation is between $4\text{Hz} < f_{\text{harmonic excitation}} < 8\text{Hz}$, then

$$\text{Response Factor} = \frac{a_{\text{peak}}}{0.007}$$

Peak acceleration criteria is normally in the range of 0.05g to 0.1g. However, lower values may be required for adjacent floor areas housing different quieter occupancies to which vibration may be transmitted.

² BACHMANN, Hugo. *Vibration Problems in Structures Practical Guidelines*. Birkhauser, Berlin, 1995.

1.1.5 Crowd Induced (Synchronous) Vertical Footfall Vibrations for Gymnasium Floors, Dance Floors, Stadium Tiers, Theatre Tiers and Balconies ³

This concerns excessive vibrations caused by groups of people dancing or jumping in unison. The vibrations may cause excessive motion resulting in panic, failure of non-structural components or in the most severe case, low cycle fatigue failure. We shall focus on whole body jumping since this result in the highest loadings and will be critical for most designs. Unlike the single person walking or running excitation, crowd induced vibrations differ in the sense that the loading is applied at the same location and also in the fact that the excitation is applied over a large spatial area simultaneously, i.e. crowd loading and not single human footfall loading.

In this LINEAR FREQUENCY DOMAIN solution, not only that the static response has to be added separately, but also the mean of the dynamic excitation has also got to be added separately as a static response. This is because the mean of the dynamically applied force is not included in the dynamic excitations. **Hence the total response in this frequency domain dynamic analysis = static response to mean of dynamic excitation + dynamic response + static response to static loads.** Clearly only the fluctuations of the dynamic excitation is required to predict the response accelerations and velocity to meet the human comfort criteria, but when an evaluation of the structural integrity is required, the dynamic response must be supplemented by the mean of the dynamic excitations, i.e. the static weight of the people, and also the static response due to the static loads on the structure.

If the natural frequency of a structure is significantly above the frequency at which repeated dynamic forces can be applied, then excessive vibration is unlikely to arise. Therefore the simplest form of check on the adequacy of a structure is to ensure that its natural frequencies are higher than the maximum frequency of effective excitation. If a minimum natural frequency of the structure is above the target value, then no further check is required. If natural frequencies are below the minimum target value, then the actual response of the structure has to be estimated; this is a more detailed and uncertain calculation. BS6399 Part 1: 1996 requires the dynamic response of structures with natural frequencies below 8.4Hz in the vertical direction and 4.0Hz in the horizontal direction to be considered explicitly. For stands occupied during pop concerts, the suggested minimum vertical frequency is intended to ensure that the second harmonic of jumping excitation is out of the frequency range. The following values are proposed in the literature.

IStructE Guide	6.0Hz
Bachmann and Ammann	7.5Hz
CEB	6.5 – 7.5Hz
NBCC (varies with mass of structure)	6.5Hz (typical)

There is also the question of whether the natural frequency should be calculated with the mass of the audience on the stand. The IStructE Guide suggests the unloaded frequency can be used but the NBCC suggests the weight of participants should be included. Ellis and Ji have observed that the mass of stationary people should be included, but that members of the audience engaged in jumping provide dynamic force only and do not contribute mass that would lead to a reduction in frequency.

There is very little guidance in the literature for swaying. Isolated measurements indicate that typical frequencies lie below 1.5Hz. International guidance of Bachmann and CEB suggest lateral frequencies should be above 2.5Hz; however, for a large crowd we assume that the effect of the second harmonic is negligible, and the minimum natural frequency should be above 1.5Hz. The IStructE does not specify a minimum natural frequency, but requires the strength of the stand to be checked for a lateral load of 7.5% of the weight of the audience, with appropriate imposed partial load factor. In a very large crowd, some reduction of DLF is expected, particularly in the higher harmonics. In addition, it becomes difficult for a large crowd to synchronize at jumping frequencies above 2.8Hz and the National Building Code of Canada suggests that the highest jumping rate to be considered is 2.75Hz. A

³ WILFORD, Michael. *An Investigation into Crowd-Induced Vertical Dynamic Loads Using Available Measurements*. Ove Arup and Partners International Ltd., 2003.

recent survey by Ji shows that almost all songs played have beat rates below 2.8Hz. The DLFs for the different activities are presented as follows according to Bachmann.

Activity	Activity Rate (Hz)	1 st Harmonic DLF	2 nd Harmonic DLF	3 rd Harmonic DLF
Jumping	Normal 2.0Hz	1.8	1.3	0.7
	Normal 3.0Hz	1.7	1.1	0.5
	High 2.0Hz	1.9	1.6	1.1
	High 3.0Hz	1.8	1.3	0.8
Dancing	2.0Hz to 3.0Hz	0.50	0.15	0.10
Hand clapping with body bouncing while standing	1.6Hz	0.17	0.10	0.04
	2.4Hz	0.38	0.12	0.02
Hand clapping	Normal 1.6Hz	0.024	0.010	0.009
	Normal 2.4Hz	0.047	0.024	0.015
	Intensive 2.0Hz	0.170	0.047	0.037
Lateral Body Swaying	Seated 0.6Hz	$DLF_{\frac{1}{2}} = 0.4$		
	Standing 0.6Hz	$DLF_{\frac{1}{2}} = 0.5$		

Other guides give the following DLF values.

Activity	First Harmonic	Second Harmonic	Third Harmonic
NBCC Dancing	0.50 (1.5 – 3Hz)	0.050 (3 – 6 Hz)	-----
NBCC Lively concert or sport event	0.40 (1.5 – 3Hz)	0.150 (3 – 6Hz)	----
NBCC Jumping/Aerobics	1.50 (1 – 2.75 Hz)	0.600 (2 – 5.5 Hz)	0.10 (3 – 8.25 Hz)
CEB 209 Dancing	0.50 (2 – 3 Hz)	0.150 (4 – 6 Hz)	0.10 (6 – 9 Hz)
CEB 209 Hand Clapping with body bouncing while standing (2.4 Hz)	0.37	0.120	0.04
Horizontal: Front to Back	0.05 (1 – 1.5 Hz)	0.040 (1.5 – 3 Hz)	Not critical
Horizontal: Side to Side	0.25 (1 – 1.5 Hz)	0.035 (1.5 – 3 Hz)	Not critical

Willford (2003) proposed the following **vertical DLFs (for jumping at pop concerts)** and lateral DLFs.

Excitation Frequency (Hz)	Vertical Dynamic Load Factor (DLF)	Coefficient of Variation
1.5 to 3.0	1.50	0.15
3.0 to 6.0	0.60	0.60
6.0 to 9.0	0.10	1.00
Excitation Frequency (Hz)	Lateral (Front to Back) Dynamic Load Factor (DLF)	Lateral (Side to Side) Dynamic Load Factor (DLF)
1.0 to 1.5	0.05	0.250
1.5 to 3.0	0.04	0.035
3.0 to 4.5	Not critical	Not critical

A forced frequency response analysis SOL 108 or SOL 111 should be carried out in MSC.NASTRAN with all nodes representing the excitations of individual people be incorporated simultaneously within one analysis. Each person induces an excitation in phase with each other, as the most critical case. Hence only one RLOAD entry is required whilst the DAREA card points to all the nodes where a single person may be jumping. This hence models the crowd induced loading and includes exactly the phase difference between the natural modes. This can be used to evaluate exactly the steady-state acceleration response, steady-state displacement response and the steady-state stress and force quantities for structural integrity.

As hand verification, a **single modal** response analysis may be performed. The modal frequency and modal mass can be obtained from a real modal analysis whilst the damping can be ascertained from a complex modal analysis or from measured data. Then, the modal force can be ascertained from

$$\text{Modal Force} = \int_{\Omega} \phi_i(\Omega) \{P(\Omega, t)\} d\Omega$$

$$\text{Amplitude of modal force, } P_{0i} = \int_{\Omega} \phi_i(\Omega) \times \text{DLF} \times \text{Static Load } d\Omega$$

The **steady-state** (i.e. **peak**) **single mode acceleration** response can then be estimated in physical coordinates as

$$a_{\text{peak}} = \phi_{\text{response}} \omega^2 \frac{P_0}{\omega_{ni}^2 M_i} \frac{1}{\sqrt{\left(1 - \omega^2 / \omega_{ni}^2\right)^2 + \left(2\zeta_i \omega / \omega_{ni}\right)^2}}$$

The following vibration acceptance criteria are proposed for severe dynamic loading cases. These are based on vertical limits proposed by Kasperski (1996) and on the relative sensitivity of humans to lateral and vertical vibration from ISO 2631.

Vertical modes:	peak acceleration to be less than 18% to 35% g maximum vertical deflection amplitude 10mm.
Horizontal modes:	peak acceleration to be less than 6% to 12% g

The lower numbers are reasonable design targets. Exceedance of the higher values (35% g vertically and 12% g horizontally) could lead to panic.

It may also be prudent to evaluate the **steady-state single mode displacement response** and hence the **steady-state single mode stress or force response** within the structure to determine the structural integrity of the structure. Note that the dynamic response must be supplemented by the mean of the dynamic excitations, i.e. the static weight of the people, and also the static response due to the static loads on the structure.

$$u_{\text{peak}} = \phi_{\text{response}} \frac{P_0}{\omega_{ni}^2 M_i} \frac{1}{\sqrt{\left(1 - \omega^2 / \omega_{ni}^2\right)^2 + \left(2\zeta_i \omega / \omega_{ni}\right)^2}}$$

1.1.6 Crowd Induced (Synchronous) Lateral Vibrations ⁴

Adequate dynamic performance of a bridge requires that

- I. the lowest lateral natural frequency should be above about **1.5Hz**, or
- II. there must be **sufficient damping** present such that synchronous lateral excitation does not occur for the crowd densities expected on the bridge. Damping ratio for stability must be

$$c > \frac{Nk}{8\pi fM}$$



The most famous instance of synchronous lateral excitation occurred on the London Millennium Bridge. The bridge structural diagram is that of a shallow suspension bridge, where the cables are as much as possible below the level of the bridge deck to free the views from the deck. Two groups of four 120mm diameter locked coil cables span from bank to bank over two river piers. The lengths of the three spans are 81m for the north span, 144m for the main span between the piers and 108m for the south span. The sag of the cable profile is 2.3m in the main span, around six times shallower than a more conventional suspension bridge structure. The tension in the cable is 22.5MN from the dead load.

It is estimated that between 80 000 and 100 000 people crossed the bridge during the first day. Analysis of video footage showed a maximum of 2000 people on the deck at any one time, resulting in a maximum density of between 1.3 and 1.5 people per square meter. Unexpected excessive lateral vibrations of the bridge occurred. The movements took place mainly on the south span, at a frequency of around 0.8Hz (the first south lateral mode), and on the central span, at frequencies of just under 0.5Hz and 1.0Hz (the first and second lateral modes respectively). More rarely, movement occurred on the north span at a frequency of just over 1.0Hz (the first north lateral mode). Excessive vibration did not occur continuously, but built up when a large number of pedestrians were on the affected spans of the bridge and died down if the number of people on the bridge reduced, or if the people stopped walking. From visual estimation of the amplitude of the movements on the south and central span, the maximum lateral acceleration experienced on the bridge was between 200 and 250 milli-g. At this level of acceleration a significant number of pedestrians began to have difficulty in walking and held onto the balustrades for support. No excessive vertical vibration was observed.

The tests on the Millennium Bridge showed that the lateral force generated by pedestrians is approximately proportional to the response of the bridge. The correlated force per person can be related to the local velocity V_{local} using a lateral walking force coefficient k as kV_{local} . The tests indicated the value of k to be approximately **300Ns/m** over the frequency range **0.5 to 1.0 Hz**.

The i^{th} person's contribution to the modal force is $\phi_i kV_{local}$. Now the local velocity is related to the modal (response) velocity by $V_{local} = \phi_i V$. Hence the i^{th} person's contribution to the modal force is $\phi_i k \phi_i V = \phi_i^2 kV$. Hence the modal excitation force generated by N people on the span is

$$F_e = \sum_{i=1}^{i=N} \left(\phi_i^2 kV \right) = kV \sum_{i=1}^{i=N} \phi_i^2$$

Since the lateral excitation force is proportional of the bridge response, there is a damping requirement for **stability**. The damping force must exceed the excitation force. The modal damping force is $2\omega cMV$. Thus if the modal damping force is to exceed the modal excitation force

⁴ DALLARD, FITZPATRICK, FLINT, LE BOURVA, LOW, RIDSDILL SMITH, WILLFORD. *The London Millennium Footbridge*. The Structural Engineer Journal Volume 79 / No. 22, Institution of Structural Engineers, London, November 2001.

$$2\omega cM > k \sum_{i=1}^{i=N} \phi_i^2$$

Rearranging and converting to frequency in Hz, the required damping for stability is

$$c > \frac{k \sum_{i=1}^{i=N} \phi_i^2}{4\pi f M}$$

This can be evaluated if the mode shape and the distribution of pedestrians is known. Assuming the people are uniformly distributed over the whole span, the number of people in the length dL is

$$dN = \frac{N}{L} dL$$

The summation can be approximated by the continuous integral

$$\sum_{i=1}^{i=N} \phi_i^2 = \int_0^L \phi^2 dN = \frac{N}{L} \int_0^L \phi^2 dL$$

Assuming the mode shape is sinusoidal and normalized to unity

$$\frac{N}{L} \int_0^L \phi^2 dL = \frac{N}{2}$$

Hence the **damping ratio required for stability**

$$c > \frac{Nk}{8\pi f M}$$

c here is the damping ratio, ζ
not the damping constant

Conversely, for a specified amount of damping, the limited number of people to avoid instability is

$$N_L = \frac{8\pi c f M}{k}$$

This formula has been derived assuming the pedestrians are **uniformly distributed over the whole span** and the **mode shape is sinusoidal** and has been **normalized to unity**.

To estimate the response, the amplitude related correlated excitation can be made equivalent to a negative damping. Damping is proportional to velocity, hence since the excitation is also proportional to the response velocity, it can be equated to a negative damping having a damping ratio

$$c_e = - \frac{k \sum_{i=1}^{i=N} \phi_i^2}{4\pi f M}$$

The negative damping varies linearly with N . At N_L the negative damping equals the actual damping, making the effective damping zero. The **effective damping** is

$$c_{\text{eff}} = c + c_e$$

$$c_{\text{eff}} = c \left(1 - \frac{N}{N_L} \right)$$

Finally, the steady-state displacement amplitude of response at resonance is approximately $1/2c_{\text{eff}}$.

Knowing the density of people expected on the bridge, the number of people N can be calculated by multiplying the density with the area of the deck. The Millennium Bridge was modified to ensure stability for a crowd density of 2.0 people / m^2 . Although tests indicate that normal walking becomes difficult at densities above about **1.7 people / m^2** . Designing for this extreme density means the bridge should always have significant positive damping, such that the response will remain comfortable. The pedestrian test results support this approach. Although lateral vibrations occurred from the onset, increasing with pedestrian numbers, the acceleration amplitudes remained acceptably small provided the number of pedestrians was less than N_L by a suitable margin.

Unless the usage of the bridge was to be greatly restricted, only two generic options to improve its performance were considered feasible. The first was to increase the stiffness of the bridge to move all its lateral natural frequencies out of the range that could be excited by the lateral footfall forces (it is believed to be possible for any frequency below about 1.3Hz, hence a minimum target of 1.5Hz seems reasonable), and the second was to increase the damping of the bridge to reduce the resonant response.

On the Millennium Bridge a total of 37 viscous dampers of viscous rate 250000 Ns/m and 4 pairs of tuned mass dampers weighing 2.5 tonnes each were installed for lateral damping. Damping of the first central lateral mode of 0.49 Hz was 20% of critical damping. Another 26 pairs of vertical tuned mass dampers weighing 1 to 3 tonnes were installed for some additional vertical damping.

MODE	Measured frequency (Hz)	Analysis frequency (Hz)	Measured modal mass (t)	Analysis modal mass (t)
CL1	0.48	0.49	128 - 130	144
CL2	0.95	0.94	145 - 148	150
CV3	1.15	1.18	155	150
CV4	1.54	1.56	140	138
CV5	1.89	1.93	135	146
CV6	2.32	2.30	135	132
SL1	0.80	0.80	160	181
NL1	1.03	1.00	Not available	113

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